HARMONIC MAPS AND HYPERSYMPLECTIC GEOMETRY

MARKUS RÖSER

Abstract. We study the hypersymplectic geometry of the moduli space of solutions to Hitchin's harmonic map equations on a G-bundle. This is the split-signature analogue of Hitchin's Higgs bundle moduli space. Due to the lack of definiteness, this moduli space is globally not well-behaved. However, we are able to construct a smooth open set consisting of solutions with small Higgs field, on which we can investigate the hypersymplectic geometry. Finally, we reinterpret our results in terms of the Riemannian geometry of the moduli space of G-connections.

Contents

1. Introduction

A *hypersymplectic manifold* is a quintuple (M^{4k}, g, I, S, T), where M is a $4k$-dimensional real manifold, g is a pseudo-Riemannian metric of signature $(2k, 2k)$ and I, S, T are skew-adjoint sections of $\mathrm{End}(TM)$ such that

$$S^2 = T^2 = \mathrm{id}_{TM} = -I^2 \qquad IS = T = -SI,$$

and

$$\nabla^g I = \nabla^g S = \nabla^g T = 0,$$

where ∇^g is the Levi-Civita connection associated with g. The skew-adjointness and the covariant constancy of I, S, T imply that

$$\omega_I = g(I-,-) \qquad \omega_S = g(S-,-) \qquad \omega_T = g(T-,-)$$

define symplectic forms on M, hence the name hypersymplectic. The above definition is reminiscent of the definition of a hyperkähler manifold and the existence

of the parallel endomorphisms I, S, T implies that the holonomy of g is contained in the non-compact Lie group $\mathrm{Sp}(2k, \mathbb{R})$, which is the split-real form of $\mathrm{Sp}(2k, \mathbb{C})$. In this way hypersymplectic manifolds are neutral-signature cousins of hyperkähler manifolds, whose holonomy is contained in the compact real form $\mathrm{Sp}(k)$. Due to the common complexification of the holonomy groups, many facts from hyperkähler geometry carry over to hypersymplectic manifolds. In particular, hypersymplectic manifolds are complex symplectic and Ricci-flat.

In the hyperkähler situation we have a whole two-sphere of compatible Kähler structures, the unit sphere in the three-dimensional real vector space spanned by I, J, K, which we may identify with the space of imaginary quaternions. Hypersymplectic structures, however, are in some sense less symmetric, since we do not only deal with in fact a whole two-sheeted hyperboloid of complex structures but also with a connected hyperboloid of product structures. Again, these hyperboloids can be thought of as spheres in the three-dimensional vector space spanned by I, S, T, which carries naturally a metric of Lorentz signature. The hyperboloids mentioned above then correspond to the subsets of spacelike, repsectively timelike, unit vectors. So there are always two different ways to look at hypersymplectic manifolds. One can study them from the point of view of complex, or in fact pseudokähler, geometry, or focus on the product structures and investigate the parakähler geometry.

A powerful tool to construct examples of hypersymplectic manifolds is the hypersymplectic quotient construction, which is an adaptation of the symplectic reduction of Marsden and Weinstein to the hypersymplectic setting. It is closely analogous to the hyperkähler quotient construction, however, the non-trivial signature of the hypersymplectic metric gives rise to pathologies not present in the hyperkähler case. The precise statement is the following.

Theorem 1.1 ([6]). *Let (M, g, I, S, T) be a hypersymplectic manifold and let G be a Lie group which acts on M preserving the hypersymplectic structure. Suppose the action is hamiltonian with respect to each of the symplectic structures ω_i with moment maps $\mu_i : M \to \mathfrak{g}^*$ for $i \in \{I, S, T\}$. Define the hypersymplectic moment map*

$$\mu = (\mu_I, \mu_S, \mu_T) : M \to \mathfrak{g}^* \otimes \mathbb{R}^3,$$

and assume that

(1) *$c \in Z(\mathfrak{g}^*) \otimes \mathbb{R}^3$ is a regular value of μ,*
(2) *G acts freely and properly on $\mu^{-1}(c)$,*
(3) *the metric g restricted to the tangent spaces to the G-orbits in $\mu^{-1}(c)$ is non-degenerate.*

Then the quotient metric on $\mu^{-1}(c)/G$ is again hypersymplectic and the symplectic forms on the quotient $\tilde{\omega}_I, \tilde{\omega}_S, \tilde{\omega}_T$ satisfy $i^*\omega_i = p^*\tilde{\omega}$ for all $i \in \{I, S, T\}$, where $p : \mu^{-1}(c) \to \mu^{-1}(c)/G$ is the projection and $i : \mu^{-1}(c) \to M$ is the inclusion map.

Conditions 1 and 2 ensure that the quotient is a smooth manifold, whereas the third condition guarantees the non-degeneracy of the symplectic forms induced by the ω_i's. In the hyperkähler case, all we have to assume is condition (2), which then implies (1) and (3). The definiteness of the hyperkähler metric is crucial in the proof and these arguments do not work anymore in our situation. We still have however that 2 and 3 together imply 1.

In most applications, we can ensure that the conditions 1 and 2 are satisfied, e.g. by changing the level c or by passing to a suitable subgroup or quotient of G. Condition 3 is more difficult to arrange and typically therefore the hypersymplectic structure on $\mu^{-1}(c)/G$ is expected to be degenerate even though the quotient manifold itself is smooth, i.e. the ω_i's are symplectic only on the complement of a *degeneracy locus*. For more details, see [6] and [2].

In this paper, we investigate a hypersymplectic structure obtained from the quotient construction in an infinite-dimensional setting, namely we study the hypersymplectic geometry of the moduli space of solutions to Hitchin's gauge theoretic equations for harmonic maps from a Riemann surface M into a compact Lie group G [5]. Historically, these equations mark the starting point of the subject of hypersymplectic geometry [6] and are given by

$$F^{\nabla} = [\Phi \wedge \Phi^*], \qquad \bar{\partial}^{\nabla}\Phi = 0, \tag{1}$$

where (∇, Φ) is a pair consisting of a G-connection ∇ on a G-vector bundle E, and a Higgs field $\Phi \in \Gamma(M, \mathfrak{g}(E)^{\mathbb{C}} \otimes K)$, where K is the canonical bundle and $\mathfrak{g}(E) \subset \mathrm{End}(E)$ is the associated bundle of Lie algebras with fibre $\mathrm{Lie}(G)$. They can be obtained from the ASD equations on $\mathbb{R}^{2,2}$ by imposing translation invariance with respect to the s and t directions, and they are the split signature analogue of Hitchin's self-duality equations on a Riemann surface [4]. The moduli space of solutions to the latter equations is known to carry a hyperkähler structure, essentially because the equations may be interpreted as the vanishing condition of a hyperkähler moment map in an infinite-dimensional setting. Since the harmonic map equations and the self-duality equations only differ by a sign in the first equation, it was natural to expect the harmonic map equations to have a moment map interpretation, too. In this way Hitchin in [6] was led to the definition of hypersymplectic structures.

Solutions to the harmonic map equations describe harmonic sections 1 of flat $G\times G$-bundles. Due to the split-signature origin of the equations, we cannot expect to have a smooth global hypersymplectic moduli space in this situation. Typically, the smoothness of gauge-theoretic moduli spaces is established by viewing the moduli space as the vanishing locus of a certain section of a vector bundle (with infinite-dimensional fibres). One then proves a vanishing theorem, which asserts that at each point of the moduli space the differential of the section is surjective, or at least of constant co-rank. It is at this point, where in addition to the ellipticity, the *definiteness* of the involved operators enters the argument. In our situation the change of signature destroys the positivity of the involve operators and we are thus not able to prove vanishing theorems which would ensure that the dimension of the moduli space does not jump. However, if we are looking for solutions with zero Higgs field, the sign change does not play a role and an argument involving the implicit function theorem enables us to produce a well-behaved neighbourhood of the moduli space of flat connections inside the moduli space of solutions to Hitchin's harmonic map equations. On this neighbourhood we can study the hypersymplectic geometry of the moduli space.

Reinterpreting the equations 1 as the equation for a geodesic segment on the space $\mathcal{A}/\mathcal{G}$ of G-connections modulo gauge transformations, whose endpoints lie on the moduli space $\mathcal{N}$ of flat connections (see [5]), we exhibit this neighbourhood as a neighbourhood of the diagonal inside $\mathcal{N} \times \mathcal{N}$. The local product structure of

this neighbourhood is the one induced by the endomorphism S of the hypersymplectic structure on the moduli space, and corresponds to assigning to a geodesic segment its endpoints. Moreover, geodesics with conjugate endpoints can be related to elements of the degeneracy locus for the hypersymplectic structure. In classical terminology of Riemannian geometry, we show that the degeneracy locus of the hypersymplectic structure is directly related to the *cut locus* of the infinite-dimensional Riemannian manifold $\mathcal{A}/\mathcal{G}$.

2. The Equations and Their Moduli Space of Solutions

Although the equations make sense for any compact structure group G, we will work in the following with $G = \mathrm{U}(n)$, i.e. consider the equations on a hermitian vector bundle E of rank n over a compact Riemann surface M. Since every compact Lie group may be embedded into $\mathrm{U}(n)$ for some n, this is not a restriction and with minor modifications our proofs work for arbitrary vector bundles with compact structure group. In this section, we study the analysis of the equations and show that inside the moduli space of solutions, there exists a smooth open set represented by solutions with small Higgs fields. The proof is a deformation argument based on the implicit function theorem. But first we set up the framework and notation.

Let M be a compact Riemann surface of genus g and let $E \to M$ be a hermitian vector bundle. Let $\mathcal{A}$ be the space of unitary connections on E. The equations are given by

$$F^{\nabla} = [\Phi \wedge \Phi^*], \qquad \bar{\partial}^{\nabla}\Phi = 0,$$

where (∇, Φ) is a pair consisting of a unitary connection ∇ and a Higgs field $\Phi \in \Omega^{1,0}(M, \mathrm{End}(E))$. We will often drop the M and write $\Omega^1(\mathfrak{u}(E)), \Gamma(\mathrm{End}(E)), \dots$ instead of $\Omega^1(M, \mathfrak{u}(E)), \Gamma(M, \mathrm{End}(E)), \dots$ in order to simplify the notation.

It will turn out to be useful to think of the pair (∇, Φ) as an element of the cotangent bundle $T^*\mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\mathrm{End}(E))$. This identification arises from the correspondence between unitary connections and holomorphic structures on E by assigning with a connection the induced $\bar{\partial}$-operator and the map

$$\Lambda : \Phi \mapsto -2i \int_M \mathrm{tr}(\Phi \wedge -),$$

which induces an isomorphism between the L^2-completions of $\Omega^{1,0}(\mathrm{End}(E))$ and $(\Omega^{0,1}(\mathrm{End}(E))^*$.

The gauge group

$$\mathcal{G} = \{u \in \Gamma(\mathrm{End}(E)) \mid u^*u = \mathrm{id}_E\} = \Gamma(\mathrm{U}(E))$$

of unitary bundle automorphisms of E acts on $T^*\mathcal{A}$ via

$$u.(\nabla, \Phi) = (\nabla + u^{-1}\mathrm{d}^{\nabla}u, u^{-1}\Phi u),$$

preserving the equations and we are interested in the moduli space of solutions to the equations modulo gauge transformations. The Lie algebra of the gauge group is given by skew-adjoint bundle endomorphisms of E, i.e. $\mathrm{Lie}(\mathcal{G}) = \Gamma(\mathfrak{u}(E)) = \{\xi \in \Gamma(\mathrm{End}(E)) \mid \xi^* = -\xi\}$, and the fundamental vector fields of the action on $T^*\mathcal{A}$ are given by

$$X^{\xi}_{(\nabla,\Phi)} = (\mathrm{d}^{\nabla}\xi, [\Phi, \xi]) =: \mathcal{D}_1(\xi).$$

That is, the linearised action is implemented by the first order differential operator

$$\mathcal{D}_1 : \mathrm{Lie}(\mathcal{G}) = \Gamma(\mathfrak{u}(E)) \to \Omega^1(\mathfrak{u}(E)) \oplus \Omega^{1,0}(\mathrm{End}(E)).$$

The action of $\mathcal{G}$ is not free on $T^*\mathcal{A}$: Any gauge transformation u of the form $e^{i\theta}\mathrm{id}_E$, where $\theta \in \mathbb{R}$, i.e. $u \in Z(\mathrm{U}(n))$, the centre of $\mathrm{U}(n)$ (note that this is *not* the centre of the gauge group $\mathcal{G}$), lies in the stabiliser of any pair (∇, Φ). So if we allow arbitrary solutions and arbitrary gauge transformations, we cannot expect to produce a well-behaved moduli space. Instead, we restrict attention to solutions with minimal stabiliser and divide the gauge group by the centre of $\mathrm{U}(n)$.

We thus say that a pair (∇, Φ) is *irreducible*, if its stabiliser is equal to $Z(\mathrm{U}(n))$ and write $T^*\mathcal{A}^*$ for the space of irreducible pairs. We define the *reduced gauge group*, which we denote by $\mathcal{G}^*$ to be

$$\mathcal{G}^* = \mathcal{G}/Z(\mathrm{U}(n)).$$

Then $\mathcal{G}^*$ acts freely on $T^*\mathcal{A}^*$ and we want to study the moduli space

$$\mathcal{M} = \{(\nabla, \Phi) \in T^*\mathcal{A}^* \; | F^\nabla - [\Phi \wedge \Phi^*] = 0 = \bar{\partial}^\nabla \Phi \; \}/\mathcal{G}^*.$$

In order to apply analytical tools, we work with the Banach space completions of $\mathcal{A}$, $T^*\mathcal{A}$ and $\mathcal{G}^*$ with respect to the L^2_k-Sobolev norm which we denote by $\mathcal{A}_k$, $T^*\mathcal{A}_k$ and $\mathcal{G}^*_k$, where on $T^*\mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\mathrm{End}(E))$ we take the direct sum of the respective Sobolev norms on each factor. We write $T^*\mathcal{A}^*_k$ for the dense open subset of irreducible pairs in $T^*\mathcal{A}_k$. The Sobolev embedding and multiplication theorems imply that for $k > 1$ the gauge group $\mathcal{G}_{k+1}$ is a smooth Hilbert Lie group and acts smoothly on $\mathcal{T}^*\mathcal{A}_k$. Note that the action of the gauge group involves derivatives of the gauge transformations, hence the different Sobolev indices.

The space $\mathcal{B}^*_k = T^*\mathcal{A}^*_k/\mathcal{G}^*_{k+1}$ of gauge equivalence classes of irreducible connections is a smooth infinite-dimensional manifold, essentially because of the existence of a local *Coulomb gauge*, which provides local slices for the action of the gauge group:

Proposition 2.1. *Let $(\nabla, \Phi) \in T^*\mathcal{A}^*_k$ be irreducible. Then there exists a constant $\epsilon(\nabla, \Phi) > 0$, such that if $(\nabla + A, \Phi + \Psi) \in T^*\mathcal{A}_{L^2_1}$ with $||A||^2_{L^4} + ||\Psi||^2_{L^4} < \epsilon$, there exists a unique gauge transformation $u \in \mathcal{G}^*_{k+1}$ such that*

$$\mathcal{D}^*_1(u.(A, \Psi)) = 0.$$

This observation is of course not new, it is also used by Hitchin in [4]. The proof of this proposition is a standard application of the implicit function theorem using the ellipticity of $\mathcal{D}^*_1\mathcal{D}_1$ and works along the lines of proposition 2.3.4 in [3].

Recall that the gauge group acts on $\mathfrak{u}(E)$-valued two-forms by conjugation, i.e. by the adjoint action Ad. Considering the principal $\mathcal{G}^*$-bundle $T^*\mathcal{A}^*_k \to \mathcal{B}^*_k$, we can form the associated vector bundle

$$\mathcal{V} = T^*\mathcal{A}^*_k \times_{\mathrm{Ad}\mathcal{G}^*_{k+1}} (\Omega^2(\mathfrak{u}(E)) \oplus \Omega^2(\mathrm{End}(E))).$$

Now we interpret the moduli space $\mathcal{M}$ as the zero locus of a section G of $\mathcal{V}$, which is defined as follows:

$$\begin{array}{rcl} G : T^*\mathcal{A}^*_k & \to & \Omega^2(\mathfrak{u}(E)) \oplus \Omega^2(\mathfrak{u}(E) \otimes \mathbb{C}) \\ G(\nabla, \Phi) & = & (F^\nabla - [\Phi \wedge \Phi^*], \bar{\partial}^\nabla \Phi). \end{array}$$

Note that $G(u.\nabla, u.\Phi) = u^{-1}(G(\nabla, \Phi))u$, i.e. G is equivariant with respect to the actions of $\mathcal{G}^*$ on $T^*\mathcal{A}^*_k$ and $\Omega^2(\mathfrak{u}(E)) \oplus \Omega^2(\mathfrak{u}(E) \otimes \mathbb{C})$, thus it descends to define a section of $\mathcal{V}$ over $\mathcal{B}^*_k$. We compute the derivative of G at a point (∇, Φ) to be

$$\mathrm{d}G_{(\nabla,\Phi)}(A, \psi) = (\mathrm{d}^\nabla A - [\Phi \wedge \psi^*] - [\psi \wedge \Phi^*], \bar{\partial}^\nabla \psi + [A^{0,1} \wedge \Phi]).$$

If (∇, Φ) solves the harmonic map equations, then the tangent space to the moduli space at this point is identified with the first cohomology space of the following deformation complex:

$$L^2_k(\mathfrak{u}(E)) \xrightarrow{\mathcal{D}_1} \Omega^1_{k-1}(\mathfrak{u}(E)) \xrightarrow{\mathrm{d}G_{(\nabla,\Phi)}} \Omega^2_{k-2}(\mathfrak{u}(E))$$

where the first map is given by the linearisation of the action, i.e. by the operator $\mathcal{D}_1$ introduced earlier and the second map is the derivative of the map G. Thus, the tangent space to the moduli space is identified with the orthogonal complement of the image of $\mathcal{D}_1$ inside the kernel of $\mathrm{d}G_{(\nabla,\Phi)}$, i.e. with $\ker(\mathcal{D}_1^* \oplus \mathrm{d}G_{(\nabla,\Phi)})$.

Note that the deformation complex is elliptic. Therefore, its cohomology groups are finite-dimensional and its Euler characteristic is given by the index of the operator $\mathcal{D}_1^* \oplus \mathrm{d}G_{(\nabla,\Phi)}$. In fact, by irreducibility we always have $\dim H^0 = 1$ for any irreducible (∇, Φ). The problem is, that we are not able to show that the cohomology group H^2 of this complex is of constant dimension independent of (∇, Φ). Therefore, the whole moduli space is not expected to be smooth.

However, for solutions with zero Higgs field, we are able to prove a vanishing theorem and construct therefore a smooth open neighbourhood of the moduli space of flat unitary connections inside the harmonic map moduli space. The precise statement is given in the following proposition.

Proposition 2.2. *Let $\nabla \in \mathcal{A}^*_k$ be an irreducible flat connection, then the image of the differential of G at $(\nabla, 0) \in \mathcal{B}^*_k$ is of constant codimension independent of the solution $(\nabla, 0)$. Moreover, its kernel has dimension $(\dim \mathrm{U}(n))4(g-1) + 4 = 4(n^2(g-1)+1)$.*

Proof. We work on a slice neighbourhood of $(\nabla, 0)$ provided by proposition 2.1, i.e. we restrict G to an appropriate ϵ-ball in $\ker \mathcal{D}_1^*$. Since we have zero Higgs field, the operator $\mathcal{D}_1$ is given by

$$\mathcal{D}_1(\xi) = (\mathrm{d}^\nabla \xi, 0) \qquad \xi \in \Gamma(\mathfrak{u}(E)).$$

The derivative $\mathrm{d}G$ simplifies to

$$\mathrm{d}G_{(\nabla,0)}(A, \psi) = (\mathrm{d}^\nabla A, \bar{\partial}^\nabla \psi).$$

Let us denote this operator by $\mathcal{D}_2(A, \psi)$. Furthermore, we have

$$\mathcal{D}_2^*(\alpha, \beta) = ((\mathrm{d}^\nabla)^* \alpha, (\bar{\partial}^\nabla)^* \beta),$$

where $(\alpha, \beta) \in \Omega^2_k(\mathfrak{u}(E)) \oplus \Omega^2_k(\mathrm{End}(E))$. In the following we drop the k to make the notation more readable but of course still work with the Sobolev completions of the relevant spaces of sections.

In order to proceed, we make use of some elliptic theory. Since the domain of G is contained in $\ker \mathcal{D}_1^*$, we have $\mathcal{D}_2 = \mathcal{D}_2 + \mathcal{D}_1^*$. So we have to check that the dimension of the kernel of the adjoint operator $\mathcal{D}_2^* + \mathcal{D}_1$ is independent of ∇.

We use the Hodge star to identify $\Omega^2(\mathfrak{u}(E)) \cong \Omega^0(\mathfrak{u}(E))$ and analogously for the complex forms. Under this identification, the operator $(\mathrm{d}^\nabla)^*$ corresponds to d^∇ and the $(\bar{\partial}^\nabla)^*$ corresponds to ∂^∇. Moreover we identify Ω^1 with $\Omega^{0,1}$ in the usual way and thus think of the operator

$$\mathcal{D}_2^* \oplus \mathcal{D}_1 : \Omega^2(\mathfrak{u}(E)) \oplus \Omega^2(\mathrm{End}(E)) \oplus \Omega^0(\mathfrak{u}(E)) \to \Omega^1(\mathfrak{u}(E)) \oplus \Omega^{1,0}(\mathrm{End}(E)),$$

after putting $\Omega^0(\mathfrak{u}(E)) \oplus \Omega^0(\mathfrak{u}(E)) \cong \Omega^0(\mathrm{End}(E))$, as the operator

$$\mathcal{D}_2^* \oplus \mathcal{D}_1 : \Omega^0(\mathrm{End}(E)) \oplus \Omega^0(\mathrm{End}(E)) \to \Omega^1(\mathfrak{u}(E)) \oplus \Omega^{1,0}(\mathrm{End}(E)),$$

given by

$$\begin{pmatrix} \mathrm{d}^{\nabla} & 0 \\ 0 & \partial^{\nabla} \end{pmatrix}.$$

That is, an element $(\xi, \eta) \in \Omega^0(\mathfrak{u}(E)) \oplus \Omega^0(\mathrm{End}(E))$ lies in the kernel if and only if

$$\mathrm{d}^{\nabla}\xi = 0 \qquad \text{and} \qquad \partial^{\nabla}\eta = 0.$$

Now by irreducibility of ∇, we immediately conclude that $\xi = 0$ in $\mathrm{Lie}(\mathcal{G}^*_{k+1})$, i.e. $\xi \in \mathfrak{u}(1)\mathrm{id}_E$. Moreover, η is also parallel as can be seen by the following integration by parts argument:

$$\begin{aligned} 0 &= ||\partial^{\nabla}\eta||^2_{L^2} \\ &= -2i \int_M \mathrm{tr}(\partial^{\nabla}\eta \wedge (\partial^{\nabla}\eta)^*) \\ &= -2i \int_M \mathrm{tr}(\partial^{\nabla}\eta \wedge -\bar{\partial}^{\nabla}(\eta^*)) \\ &= -2i \int_M \bar{\partial}\mathrm{tr}((\partial^{\nabla}\eta)\eta^*) - \mathrm{tr}((\bar{\partial}^{\nabla}\partial^{\nabla}\eta)\eta^*) \\ &= -2i \int_M \mathrm{tr}(-(\partial^{\nabla}\bar{\partial}^{\nabla}\eta)\eta^*) \qquad (\text{as } \nabla \text{ is flat}) \\ &= -2i \int_M \mathrm{tr}(\bar{\partial}^{\nabla}\eta \wedge (\bar{\partial}^{\nabla}\eta)^*) \\ &= ||\bar{\partial}^{\nabla}\eta||^2_{L^2}. \end{aligned}$$

Thus, since ∇ is assumed to be irreducible, it follows that $\xi \in i\mathbb{R}\mathrm{id}_E$ and $\eta \in \mathbb{C}\mathrm{id}_E$. In other words, H^2 is of real dimension 3.

The statement about the dimension of the kernel of $\mathrm{d}G$ is obtained from the Atiyah-Singer-Index theorem. The details can be found in [4], section 5. This makes sense, since in the case $\Phi = 0$ both complexes here and in [4] reduce to the same elliptic complex. Note however, that from the above discussion we have $H^0 = 1$ and $H^2 = 3$ in the deformation complex. Thus the proof of the proposition is complete. □

Corollary 2.3. *Let ∇ be an irreducible flat connection, then on a sufficiently small neighbourhood of ∇ in $\mathcal{A}$ there exists a $4(n^2(g-1)+1)$ dimensional family of solutions to the harmonic map equations.*

3. The Hypersymplectic Geometry of the Moduli Space

3.1. **The Moment Map Interpretation of the Equations.** We now apply the hypersymplectic quotient construction in an infinite-dimensional setting in order to study the geometry of the moduli space of solutions to Hitchin's harmonic map equations.

On $T^*\mathcal{A}$ we have a natural complex structure I induced by the Hodge-star operator acting on one-forms on M. Together with the L^2 inner product we get an indefinite Kähler structure on $T^*\mathcal{A}$. Under the identification $\Omega^{1,0}(\mathrm{End}(E)) \cong (\Omega^{0,1}(\mathrm{End}(E)))^*$ discussed earlier, the complex structure induced by the Hodge-star operator is just multiplication by i, that is,

$$I(A, \Phi) = (iA, i\Phi).$$

The indefinite metric reads

$$g((A,\Phi),(B,\Psi)) = \mathrm{Re}\left(2i\int_M \mathrm{tr}(A^*\wedge B - \Phi\wedge\Psi^*)\right).$$

Like on any complex cotangent bundle, we also have the canonical holomorphic symplectic form $\omega_I^{\mathbb{C}}$ given by

$$\omega_I^{\mathbb{C}}((A,\Phi),(B,\Psi)) = \Lambda(\Psi)(A) - \Lambda(\Phi)(B) = 2i\int_M \mathrm{tr}(\Phi\wedge B - \Psi\wedge A).$$

This clearly has type $(2,0)$ with respect to the complex structure I. We now define endomorphisms S and T of $T(T^*\mathcal{A})$ by taking the real and imaginary parts of $\omega_I^{\mathbb{C}}$. That is, we write

$$\omega_I^{\mathbb{C}} = g(S-,-) + ig(T-,-).$$

Noting that $\mathrm{Re}(\mathrm{tr}(\Phi\wedge A)) = \mathrm{Re}(\mathrm{tr}(A^*\wedge\Phi^*))$ a direct calculation shows that S and T are given by

$$S(A,\Phi) = (\Phi^*, A^*) \qquad T(A,\Phi) = (i\Phi^*, iA^*) = IS(A,\Phi).$$

The action of the gauge group preserves the above flat hypersymplectic structure on $T^*\mathcal{A}$, and we show now that it admits a hypersymplectic moment map. We only calculate the moment map μ_I explicitly and then give the other two moment maps without proof in order to avoid repetition as the calculations are very similar. Recall that the fundamental vector fields of the action are given by

$$X^{\xi}_{(\nabla,\Phi)} = \mathcal{D}_1(\xi) = (\mathrm{d}^{\nabla}\xi, [\Phi,\xi]).$$

Now using bi-invariance of the trace inner product and Stokes' theorem, we compute analogously to the Atiyah-Bott calculation (see [1], and also [4])

$$\begin{aligned}
\omega_I(X_\xi,(A,\Psi)) &= \int_M \mathrm{tr}(\mathrm{d}^{\nabla}\xi\wedge A) - \int_M \mathrm{tr}([\Phi-\Phi^*,\xi]\wedge(\Psi-\Psi^*)) \\
&= \int_M \mathrm{tr}((\mathrm{d}^{\nabla}A - [\Psi\wedge\Phi^*] - [\Phi\wedge\Psi^*])\xi) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\int_M \mathrm{tr}((F^{\nabla+tA} - [\Phi+t\Psi\wedge\Phi^*+t\Psi^*])\xi).
\end{aligned}$$

Thus, under the identification $\mathrm{Lie}(\mathcal{G})^* \cong \Omega^2(\mathfrak{u}(E))$ via the pairing $(\xi,\alpha)\in\mathrm{Lie}(\mathcal{G})\times\Omega^2(\mathfrak{u}(E))\mapsto\int_M\mathrm{tr}(\alpha\xi)$, we see that the moment map is given by

$$\mu_I(\nabla,\Phi) = \mathrm{F}^{\nabla} - [\Phi\wedge\Phi^*].$$

A similar calculation yields that

$$(\mu_S + i\mu_T)(\nabla,\Phi) = \bar\partial^{\nabla}\Phi,$$

and so the vanishing of the moment map is indeed given by the harmonic map equations,

$$\begin{aligned}
\mu_I^{\mathbb{C}}(\nabla,\Phi) &= \bar\partial^{\nabla}\Phi = 0, \\
\mu_I(\nabla,\Phi) &= F^{\nabla} - [\Phi\wedge\Phi^*] = 0.
\end{aligned}$$

Thus, formally, the moduli space can be interpreted as a hypersymplectic quotient in an infinite-dimensional setting, and so we move now on to study the hypersymplectic geometry of the smooth open set constructed in the previous section.

3.2. **Degeneracies.** We start by characterising the degeneracy locus.

Proposition 3.1. *An element $(\nabla, \Phi) \in T^*\mathcal{A}$ lies in the degeneracy locus if and only if the kernel of $\mathcal{D}_1^\dagger \mathcal{D}_1$ is non-zero. Here $\mathcal{D}_1 : \Gamma(\mathfrak{u}(E)) \to \Omega^1(\mathfrak{u}(E)) \oplus \Omega^1(\mathfrak{u}(E))$ is the operator defined by the infinitesimal gauge action and $\dagger$ denotes the adjoint taken with respect to the split signature metric on $\Omega^1(\mathfrak{u}(E)) \oplus \Omega^1(\mathfrak{u}(E))$. This is an elliptic operator, given by*

$$\mathcal{D}_1^\dagger \mathcal{D}_1(\xi) = (\mathrm{d}^\nabla)^* \mathrm{d}^\nabla \xi + *[\phi \wedge *[\phi, \xi]],$$

where $\phi = \Phi - \Phi^*$.

Proof. We want to compute the intersection of the tangent space to a gauge orbit with its orthogonal complement. We work with real co-ordinates, $\phi = \Phi - \Phi^*$. The fundamental vector fields of the gauge action are given by

$$X^\xi_{(\nabla,\phi)} = (\mathrm{d}^\nabla \xi, [\phi, \xi]) = (\mathrm{d}^\nabla \xi, \mathrm{ad}(\phi)(\xi)) = \mathcal{D}_1 \xi.$$

We compute the adjoint of $\mathcal{D}_1$ with respect to the neutral inner product defined above. Let $(A, \psi) \in \Omega^1(\mathfrak{u}(E)) \oplus \Omega^1(\mathfrak{u}(E))$. Then the adjoint is characterised by the property

$$g(\mathcal{D}_1 \xi, (A, \psi)) = \langle \xi, \mathcal{D}_1^*(A, \psi) \rangle_{L^2}.$$

The only thing we actually have to compute is the adjoint of $\mathrm{ad}(\phi)(\xi)$ with respect to the ordinary L^2 inner product. Let $A \in \Omega^1(\mathfrak{u}(E))$.

$$\begin{aligned}
\langle \mathrm{ad}(\phi)(\xi), A \rangle_{L^2} &= -\int_M \mathrm{tr}([\phi, \xi] \wedge *A) \\
&= -\int_M \mathrm{tr}((\phi\xi - \xi\phi) \wedge *A) \\
&= -\int_M \mathrm{tr}((\xi(-*A \wedge \phi - \phi \wedge *A)) \\
&= -\int_M \mathrm{tr}(\xi * (-*[\phi \wedge *A])) \\
&= \langle \xi, -*[\phi \wedge *A] \rangle_{L^2}.
\end{aligned}$$

So the adjoint is given by $\mathrm{ad}(\phi)^*(A) = -*\mathrm{ad}(\phi)(*A)$. With this we now compute for $\xi, \eta \in \Gamma(\mathfrak{u}(E))$:

$$\begin{aligned}
g(\mathcal{D}_1\xi, \mathcal{D}_1\eta) &= \langle \mathrm{d}^\nabla \xi, \mathrm{d}^\nabla \eta \rangle_{L^2} - \langle \mathrm{ad}(\phi)(\xi), \mathrm{ad}(\phi)(\eta) \rangle_{L^2} \\
&= \langle (\mathrm{d}^\nabla)^* \mathrm{d}^\nabla \xi, \eta \rangle_{L^2} - \langle (\mathrm{ad}(\phi))^* \mathrm{ad}(\phi)(\xi), \eta \rangle_{L^2} \\
&= \langle (\mathrm{d}^\nabla)^* \mathrm{d}^\nabla \xi + *[\phi \wedge *[\phi, \xi]], \eta \rangle_{L^2}.
\end{aligned}$$

Thus, we conclude that $\mathcal{D}_1\xi$ lies in the orthogonal complement of the tangent space to the gauge orbit through (∇, ϕ), i.e. in the kernel of $\mathcal{D}_1^\dagger$ if and only if

$$(\mathrm{d}^\nabla)^* \mathrm{d}^\nabla \xi + *[\phi \wedge *[\phi, \xi]] = 0,$$

as asserted. □

Since ϕ is skew-adjoint, we get that the operator $*[\phi \wedge *[\phi, -]]$ is self-adjoint with non-positive eigenvalues. Hence, the self-adjoint elliptic operator $(\mathrm{d}^\nabla)^* \mathrm{d}^\nabla \xi + *[\phi \wedge *[\phi, \xi]]$, being the sum of a non-negative and a non-positive operator, is in general not positive and might a priori have a non-trivial kernel.

3.3. **Product Structures.** On any hypersymplectic manifold (M, g, I, S, T) there is a circle of product structures orthogonal to I given by

$$S_\theta = \cos\theta S - \sin\theta T, \qquad T_\theta = \sin\theta S + \cos\theta T,$$

or in more compact notation

$$S_\theta + iT_\theta = e^{i\theta}(S + iT).$$

These product structures are integrable, therefore, for each θ, a hypersymplectic manifold is locally a product of integral submanifolds associated with the distributions given by the ± 1-eigenspaces of S_θ (analogously for T_θ). In our situation, the hypersymplectic manifold in question is the cotangent bundle of the space of unitary connections and we obtain

$$S_\theta = \cos\theta S - \sin\theta T = \begin{pmatrix} 0 & -\cos\theta - \sin\theta * \\ -\cos\theta + \sin\theta * & 0 \end{pmatrix},$$

where $*$ is, as usual, the Hodge star operator acting on one-forms. Since $*$ squares to -1 on one-forms, a suggestive short-hand notation is

$$S_\theta = \begin{pmatrix} 0 & -e^{*\theta} \\ -e^{-*\theta} & 0 \end{pmatrix}.$$

Given $\theta \in \mathbb{R}$ and a solution $(\nabla, \Phi)$of the harmonic map equations, we can associate with it a pair of connections $(\nabla_\theta^+, \nabla_\theta^-)$ given by

$$\nabla_\theta^\pm = \nabla \pm e^{\theta *}(\Phi - \Phi^*).$$

As a consequence of the harmonic map equations, these connections are *flat*. It turns out that this map really implements the local product structure S_θ. In fact, an easy manipulation shows that, identifying $T^*\mathcal{A} \cong \mathcal{A} \times \Omega^1(\mathfrak{u}(E))$, the harmonic map equations for a pair (∇, ϕ) can be written as

$$\begin{aligned} F^{\nabla+\phi} &= 0 \\ F^{\nabla-\phi} &= 0 \\ (\mathrm{d}^\nabla)^*\phi &= 0. \end{aligned}$$

Writing $\phi = \Phi - \Phi^*$ we obtain the original form of the equations. In other words, the Higgs field Φ is the $(1,0)$-part of ϕ.

Proposition 3.2. *The map*

$$P_\theta : T^*\mathcal{A} \to \mathcal{A} \times \mathcal{A} \qquad (\nabla, \phi) \mapsto (\nabla + e^{\theta *}\phi, \nabla - e^{\theta *}\phi)$$

identifies $(T^*\mathcal{A}, S_\theta)$ *with* $(\mathcal{A} \times \mathcal{A}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})$ *as paracomplex manifolds.*

Proof. Let us write $\mathbf{s}$ for the paracomplex structure on $\mathcal{A} \times \mathcal{A}$. We take the derivative of P and show that $\mathbf{s} \circ \mathrm{d}P = \mathrm{d}P \circ S_\theta$. The derivative of P_θ is given by

$$\mathrm{d}P_\theta = \begin{pmatrix} 1 & e^{\theta *} \\ 1 & -e^{\theta *} \end{pmatrix}.$$

Now

$$\mathrm{d}P_\theta \circ S_\theta = \begin{pmatrix} 1 & e^{\theta *} \\ 1 & -e^{\theta *} \end{pmatrix} \begin{pmatrix} 0 & -e^{*\theta} \\ -e^{-*\theta} & 0 \end{pmatrix} = \begin{pmatrix} -1 & -e^{*\theta} \\ 1 & -e^{\theta *} \end{pmatrix}.$$

On the other hand

$$\mathbf{s} \circ \mathrm{d}P_\theta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{\theta *} \\ 1 & -e^{\theta *} \end{pmatrix} = \begin{pmatrix} -1 & -e^{\theta *} \\ 1 & -e^{\theta *} \end{pmatrix}.$$

The inverse of P_θ is given by

$$P_\theta^{-1}(\nabla_1, \nabla_2) = (\frac{1}{2}(\nabla_1 + \nabla_2), \frac{e^{-\theta *}}{2}(\nabla_1 - \nabla_2)).$$

□

For any θ this map associates with a solution of the harmonic map equations a pair of flat connections. Moreover, this map is gauge equivariant, so it descends to a map on the respective moduli spaces.

However, the map induced by P_θ in general does not have to be injective on the moduli space. It may happen that different solutions to the harmonic map equations give rise to gauge equivalent pairs of flat connections. However, it will turn out that on our open set constructed in the previous section, it actually is injective. We postpone the proof of this statement to the final section, see corollary 4.4.

We note that the singular points in the image of the map P_θ have to come from the degeneracy locus:

Lemma 3.3. *Let (∇, ϕ) be a solution to the harmonic map equations. If $\nabla_\theta^+ = \nabla + e^{\theta * \phi}$ or $\nabla_\theta^- = \nabla - e^{\theta * \phi}$ are reducible, then the solution (∇, ϕ) lies in the degeneracy locus.*

Proof. We prove the lemma in the case $\theta = 0$, the case of general θ is treated analogously. Suppose ∇^+ is reducible. Then there exists a section $\xi \in \Gamma(\mathfrak{u}(E))$, which is not a constant multiple of the identity, such that

$$0 = \mathrm{d}^{\nabla^+}\xi = \mathrm{d}^{\nabla}\xi + [\phi, \xi].$$

Now consider

$$\begin{aligned} 0 &= (\mathrm{d}^{\nabla^-})^* \mathrm{d}^{\nabla^+}\xi \\ &= - * \mathrm{d}^{\nabla^-} * \mathrm{d}^{\nabla^+}\xi \\ &= - * (\mathrm{d}^{\nabla} * \mathrm{d}^{\nabla^+}\xi + [\phi \wedge * \mathrm{d}^{\nabla^+}\xi]) \\ &= - * (\mathrm{d}^{\nabla} * \mathrm{d}^{\nabla}\xi - [*\phi \wedge \mathrm{d}^{\nabla}\xi] - [\phi \wedge * \mathrm{d}^{\nabla}\xi] - [\phi \wedge *[\phi, \xi]]) \\ &= (\mathrm{d}^{\nabla})^* \mathrm{d}^{\nabla}\phi + *[\phi \wedge *[\phi, \xi]], \end{aligned}$$

where in the last line we used the equation $\mathrm{d}^{\nabla}\xi = -[\phi, \xi]$ and the Jacobi identity. □

It turns out that all product structures on the circle orthogonal to I are equivalent.

Proposition 3.4. *The circle action $(\nabla, \Phi) \mapsto (\nabla, e^{i\alpha}\Phi)$ induces a paraholomorphic diffeomorphism $(T^*\mathcal{A}, S_\theta) \cong (T^*\mathcal{A}, S_{\theta+\alpha})$.*

Proof. Let $\tau : \Omega^1(\mathrm{End}(E)) \to \Omega^1(\mathrm{End}(E))$ be the transposition map (or more invariantly the anti-linear involution induced by minus the Cartan involution) $\tau(\Phi) = \Phi^*$. The product structure S_θ may then be written as

$$S_\theta = \begin{pmatrix} 0 & e^{i\theta}\tau \\ e^{i\theta}\tau & 0 \end{pmatrix}.$$

For fixed α, the derivative of the map $(\nabla, \Phi) \mapsto (\nabla, e^{i\alpha}\Phi)$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}.$$

The proof is finished by a direct calculation:

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} 0 & e^{i\theta}\tau \\ e^{i\theta}\tau & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i\theta+\alpha}\tau \\ e^{i\theta+\alpha}\tau & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}.$$

□

3.4. **Complex Structures.** Recall that on any hypersymplectic manifold, we have a two-sheeted hyperboloid of complex structures, which we may parametrise by $\zeta \in \mathbb{CP}^1 \setminus \{|\zeta| = 1\}$:

$$I_\zeta = \frac{1}{1-|\zeta|^2}\left((1+|\zeta|^2)I + (\zeta+\bar{\zeta})S + i(\zeta-\bar{\zeta})T\right).$$

Note that in this notation $I = I_0$. With respect to the complex structure I_0, we have already seen that the map assigning to a unitary connection and a Higgs field the associated $\bar{\partial}$-operator and the $(1,0)$-component of the Higgs field is biholomorphic. It identifies $(\mathcal{A} \times \Omega^1(\mathfrak{u}(E)), I_0)$ with the holomorphic cotangent bundle $T^*\mathcal{A}$ of the space of $\bar{\partial}$-operators. But how about the other complex structures?

Definition 1. Let $\lambda \in \mathbb{C}^*$. A *partial λ-connection* on a hermitian vector bundle (E, h) is a $\mathbb{C}$-linear map

$$\nabla^\lambda : \Gamma(E) \to \Omega^{1,0}(E),$$

such that

$$\nabla^\lambda(fs) = \lambda\partial f \otimes s + f\nabla^\lambda s,$$

for all $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

To our knowledge, the definition of a λ-connection is due to Deligne and appeared first in Simpson's work on non-abelian Hodge theory, see for example [8].

We denote by $\mathcal{A}^\lambda$ the set of partial λ-connections on E, which is an affine space modelled on $\Omega^{1,0}(\mathrm{End}(E))$. We also observe that if $\lambda = 0$, we may think of a 0-connection as just a $\mathrm{End}(E)$-valued $(1,0)$-form. The complex gauge group acts on $\mathcal{A}^\lambda$ in a natural way by conjugation.

Proposition 3.5. *Let $\zeta \in \mathbb{C}$ with $|\zeta| \neq 1$. The map*

$$\begin{array}{rcl} F_\zeta : (T^*\mathcal{A}, I_\zeta) & \to & (\mathcal{A} \times \mathcal{A}^{-i\bar{\zeta}}, i \oplus i) \\ (\bar{\partial}^\nabla, \Phi) & \mapsto & (\bar{\partial}^\nabla - i\bar{\zeta}\Phi^*, -i\bar{\zeta}\partial^\nabla + \Phi) \end{array}$$

is a $\mathcal{G}$-equivariant holomorphic diffeomorphism.

Proof. Let τ be the transposition map introduced in the proof of proposition 3.4. Then we may write $I_\zeta \in \mathrm{End}(\Omega^{1,0}(\mathrm{End}(E)) \oplus \Omega^{1,0}(\mathrm{End}(E)))$ schematically as the two by two matrix

$$I_\zeta = \frac{1}{1-|\zeta|^2}\begin{pmatrix} (1+|\zeta|^2)i & 2\bar{\zeta}\tau \\ 2\bar{\zeta}\tau & (1+|\zeta|^2)i \end{pmatrix}.$$

The derivative of F_ζ is given by

$$\mathrm{d}F_\zeta(A, \Phi) = \begin{pmatrix} 1 & -i\bar{\zeta}\tau \\ -i\bar{\zeta}\tau & 1 \end{pmatrix}.$$

Now a direct computation, keeping in mind that τ is conjugate-linear, gives

$$\mathrm{d}F_\zeta \circ I_\zeta = i\mathrm{d}F_\zeta.$$

The equivariance is clear. $\square$

Thinking of 0-connections as Higgs fields, we see that the map F_ζ is a direct generalisation of the map F_0 given in these co-ordinates by the identity.

In analogy to the case of Higgs bundles, it turns out that apart from $\pm I$ all other complex structures are equivalent.

Proposition 3.6. *The complex structures I_ζ, where $\zeta \neq 0, \infty$, are all equivalent.*

Proof. Let $\lambda, \zeta \in \mathbb{C}^*$. Then the map

$$\mathcal{A} \times \mathcal{A}^\lambda \to \mathcal{A} \times \mathcal{A}^\zeta,$$

given by the identity on the first factor and multiplication by $\zeta\lambda^{-1}$ on the second factor, gives the desired biholomorphism. $\square$

A rearrangement argument shows that given two solutions with sufficiently small Higgs fields, which are gauge equivalent by a complex gauge transformation, they already are by unitary ones.

Proposition 3.7 (Local Uniqueness)**.** *Let (∇_i, Φ_i) be two solutions to the harmonic map equations defined on a hermitian vector bundle E. Suppose that there exists a complex gauge transformation $u \in L^2_k(\mathrm{GL}(E))$ such that*

$$(\bar\partial^{\nabla_1}, \Phi_1) = u.(\bar\partial^{\nabla_2}, \Phi_2).$$

Then (∇_1, Φ_1) and (∇_2, Φ_2) are gauge equivalent by a unitary gauge transformation, provided $\Phi = -\Phi_1^t \otimes 1 + 1 \otimes \Phi_2$ satisfies

$$||\Phi \wedge \Phi^*||^2_{L^2} < \lambda_1(\nabla),$$

where ∇ is the induced connection on $\mathrm{E}^* \otimes E \cong \mathrm{End}(E)$ with ∇_1 acting on E^* and ∇_2 acting on E and $\lambda_1(\nabla)$ denotes the first non-zero eigenvalue of its associated Laplacian acting on sections of $\mathrm{End}(E)$*.*

Proof. It is straight-forward to check that (∇, Φ) satisfies the harmonic map equations. The map u being a complex gauge transformation transforming (∇_2, Φ_2) into (∇_1, Φ_1), means that

$$\bar\partial^\nabla u = 0,$$

if we view u as a section of $\mathrm{End}(E)$.

Moreover, as Φ_1 acts on E^* via $\Phi(\alpha)(v) = -\alpha(\Phi(v))$, it follows that $\Phi u = -u\Phi_1 + \Phi_2 u = 0$, since $u^{-1}\Phi_2 u = \Phi_1$.

Since the Laplacian Δ^∇ associated with ∇ is elliptic, self-adjoint and positive, it follows from the compactness of M that $L^2(\mathrm{End}(E))$ decomposes into an orthogonal direct sum of its (finite-dimensional) eigenspaces. Decompose $u = u_0 + u_\perp$, with u_0 the orthogonal projection onto $\ker(\Delta^\nabla) = \ker(\mathrm{d}^\nabla)$ and $u_\perp = u - u_0$. Let λ_1 be the smallest non-zero eigenvalue of Δ^∇. Now we apply a Weitzenböck argument.

$$\begin{aligned}
||\mathrm{d}^\nabla u||^2_{L^2} &= ||\mathrm{d}^\nabla u_\perp||^2_{L^2} \\
&= ||\partial^\nabla u_\perp||^2_{L^2} \\
&= \int_M \mathrm{tr}(F^\nabla u_\perp \wedge * u_\perp) \\
&= \int_M \mathrm{tr}([\Phi \wedge \Phi^*] u_\perp \wedge * u_\perp) \\
&= \int_M \mathrm{tr}(\Phi \wedge \Phi^* u_\perp \wedge * u_\perp) \\
&\leq ||\Phi \wedge \Phi^*||^2_{L^2} ||u_\perp||^2_{L^2} \\
&< \lambda_1 ||u_\perp||^2_{L^2}.
\end{aligned}$$

On the other hand, we have that

$$\lambda_1 ||u_\perp||^2_{L^2} \leq \langle \Delta^\nabla u_\perp, u_\perp \rangle_{L^2} = ||\mathrm{d}^\nabla u_\perp||^2_{L^2} < \lambda_1 ||u_\perp||^2_{L^2}.$$

So we conclude that $u_\perp = 0$ and hence u is parallel with respect to ∇ and moreover

$$0 = ||\mathrm{d}^\nabla u||^2_{L^2} = \langle \Phi \wedge \Phi^* u, u \rangle = ||\Phi^* u||^2,$$

so $\Phi^* u = 0$. Now we define the unitary gauge transformation

$$\tilde{u} = u(u^* u)^{-\frac{1}{2}}.$$

Then $\tilde{u}$ is also parallel and hence gauges ∇_2 to ∇_1. Furthermore, since $\Phi u = 0 = \Phi^* u$, it follows that $\Phi u^* = 0$ and hence $\Phi \tilde{u} = 0$, i.e

$$\tilde{u}^{-1} \Phi_2 \tilde{u} = \Phi_1.$$

□

Remark 3.8. This proposition shows that a small neighbourhood of the moduli space of flat connections in the moduli space of solutions to the harmonic map equations may be identified with an appropriate moduli space of $\lambda-$connections modulo complex gauge transformations.

4. The Riemannian Geometry of the Moduli Space of Connections

In this section we investigate the equations from an alternative point of view, which originates in the following observation [5].

Let $k > 1$ and consider the space $\mathcal{A}^*_k$ of irreducible unitary connections of Sobolev class k on the hermitian vector bundle E. As we have seen, this is an infinite-dimensional affine space modelled on $\Omega^1_k(\mathfrak{u}(E))$. Equipped with the L^2 inner product, we may view this as a flat infinite-dimensional Hilbert manifold. The Hilbert Lie group $\mathcal{G}^*_{k+1}$ of reduced gauge transformations acts freely on $\mathcal{A}^*_k$ by isometries. If we put the quotient metric on $\mathcal{A}^*_k/\mathcal{G}^*_{k+1}$, then the projection $\mathcal{A}^*_k \to \mathcal{A}^*_k/\mathcal{G}^*_{k+1}$ becomes a Riemannian submersion. It then turns out that the harmonic map equations can be given a natural interpretation in the terms of the Riemannian geometry of $\mathcal{A}^*_k/\mathcal{G}^*_{k+1}$.

Since $\mathcal{A}^*_k$ is just an affine space, geodesics are given by straight lines. Now since $\mathcal{A}^*_k \to \mathcal{A}^*_k/\mathcal{G}^*_{k+1}$ is a Riemannian submersion, geodesics on the base can, at least locally, be lifted to horizontal geodesics on $\mathcal{A}^*_k$, i.e. geodesics orthogonal to the $\mathcal{G}^*_{k+1}$-orbits. At a point $\nabla \in \mathcal{A}^*_k$, the tangent space to the $\mathcal{G}^*_{k+1}$-orbit is given by

the image of $\mathrm{d}^\nabla : L^2_{k+1}(\mathfrak{u}(E)) \to \Omega^1_k(\mathfrak{u}(E))$. Thus, a geodesic $\gamma(t) = \nabla + t\phi$ is horizontal at ∇ if and only if $(\mathrm{d}^\nabla)^*\phi = 0$. Notice that since $[\phi \wedge *\phi] = 0$, it then automatically follows that $(\mathrm{d}^{\nabla+t\phi})^*\phi = 0$, and the geodesic is in fact horizontal for all t. Therefore, in order to specify a horizontal geodesic, we need to fix a point ∇ on the geodesic and a horizontal direction vector $\phi \in \ker(\mathrm{d}^\nabla)^*$. In other words, we can think of the space $T^*\mathcal{A}^*_k$ as the space of horizontal geodesic segments on $\mathcal{A}^*_k$.

Decomposing ϕ into its $(1,0)$ and $(0,1)$ parts, i.e. writing $\phi = \Phi - \Phi^*$ with $\Phi \in \Omega^{1,0}(\mathrm{End}(E))$, the harmonic map equations are equivalent to the system

$$\begin{aligned} F^{\nabla+\phi} &= 0 \\ F^{\nabla-\phi} &= 0 \\ (\mathrm{d}^\nabla)^*\phi &= 0. \end{aligned}$$

In other words, we may interpret the harmonic map equations as the equation of a geodesic on $\mathcal{A}^*_k/\mathcal{G}^*_{k+1}$ whose endpoints $\nabla \pm \phi$ are contained in the moduli space of flat unitary connections, which we denote by $\mathcal{N}_k$. We summarise this discussion in the following proposition.

Proposition 4.1 ([5]). *Solutions to the harmonic map equations modulo gauge equivalence are in one-to-one correspondence with geodesics on the moduli space $\mathcal{A}^*_k/\mathcal{G}^*_{k+1}$ of irreducible unitary connections whose endpoints are contained in the moduli space $\mathcal{N}_k$ of flat connections.*

4.1. **An Existence Theorem.** In this setting the existence of solutions with sufficiently small Higgs field follows rather easily. Observe that since $(\mathrm{d}^{\nabla-\phi})^*\phi = 0$, the connection $\nabla^+ = \nabla + \phi$ is in Coulomb gauge with respect to $\nabla^- = \nabla - \phi$ (and vice versa of course). Thus, the existence of solutions with sufficiently small Higgs field follows from the existence of a local Coulomb gauge.

Proposition 4.2. *Let $k > 1$ and let $\nabla \in \mathcal{A}^*_k$ be irreducible. Then there exists a constant $\epsilon(\nabla) > 0$, such that if $\nabla + A \in \mathcal{A}_k$ with $A \in \Omega^1(\mathfrak{u}(E))$ satisfies $||A||^2_{L^4} < \epsilon$, there exists a unique gauge transformation $u \in \mathcal{G}^*_{k+1}$ such that*

$$(\mathrm{d}^\nabla)^*(u^{-1}Au + u^{-1}\mathrm{d}^\nabla u) = 0.$$

Together with the elementary fact that flatness is a gauge-invariant condition, this proposition proves the existence of short geodesics, i.e. solutions to the harmonic map equations with small Higgs field. Moreover, the product structure S on $T^*\mathcal{A}^*_k$ has a natural interpretation, as it assigns to a geodesic segment linking ∇^- to ∇^+ its endpoints. Our aim is now to prove that this map is injective on sufficiently short geodesics, i.e. solutions with sufficiently small Higgs fields. The following theorem asserts, that any connection has a neighbourhood in which any two points can be linked by a unique horizontal geodesic. Thus, in this neighbourhood geodesic segments are uniquely determined by their endpoints.

Theorem 4.3. *Let $\nabla \in \mathcal{A}^*_{k-1}$ be an irreducible connection and let $\nabla_i = \nabla + A_i$ for $i = 1, 2$ be two connections with ∇_1 in Coulomb gauge relative to ∇, i.e. $(\mathrm{d}^\nabla)^*A_1 = 0$. Then there exists a constant $C > 0$ depending on ∇, but which is independent of A_1 and A_2 such that if $||A_1||_{k-1} < C$ there exists a gauge transformation $u \in \mathcal{G}^*_k$ such that $u.\nabla_2$ is in Coulomb gauge with respect to ∇_1, provided the norm of A_2 is sufficiently small.*

Proof. The proof is similar to the proof of the existence of a local Coulomb gauge and uses the implicit function theorem. The equation we want to solve for a small $\xi \in L^2_k(\mathfrak{u}(E))$ is

$$(\mathrm{d}^{\nabla_1})^*(\nabla_1 - \exp\xi.\nabla_2) = 0.$$

In terms of ∇ and the connections matrices A_i this reads

$$(\mathrm{d}^{\nabla_1})^*(A_1 - A_2 - (\exp -\xi)\mathrm{d}^{\nabla_2}\exp\xi) = 0.$$

We view this as a map between Sobolev spaces:

$$F(A_2, \xi) = (\mathrm{d}^{\nabla_1})^*(A_1 - A_2 - (\exp -\xi)\mathrm{d}^{\nabla + A_2}\exp\xi) = 0,$$

where

$$F : \Omega^1_{k-1}(\mathfrak{u}(E)) \times L^2_k(\mathfrak{u}(E)) \to \mathrm{Im}((\mathrm{d}^{\nabla_1})^*) \subset L^2_{k-2}(\mathfrak{u}(E)).$$

By assumption, $F(0,0) = 0$. We have to show that the partial derivative of F with respect to ξ at $(A_2, \xi) = (0,0)$ is surjective. A computation shows that this is given by

$$D_2F(\eta) = (\mathrm{d}^{\nabla_1})^*\mathrm{d}^{\nabla}\eta.$$

We show that D_2F is surjective if the norm of A_1 is sufficiently small. Suppose it is not surjective, then we can find $\chi \in \mathrm{Im}((\mathrm{d}^{\nabla_1})^*)$ which is orthogonal to the image of $(\mathrm{d}^{\nabla_1})^*\mathrm{d}^{\nabla}$, that is, for all η we have

$$\langle (\mathrm{d}^{\nabla_1})^*\mathrm{d}^{\nabla}\eta, \chi\rangle_{L^2} = 0.$$

Put $\eta = \chi$ and compute

$$0 = \langle \mathrm{d}^{\nabla}\chi, \mathrm{d}^{\nabla + A_1}\chi\rangle_{L^2} = ||\mathrm{d}^{\nabla}\chi||^2_{L^2} + \langle \mathrm{d}^{\nabla}\chi, [A_1, \chi]\rangle_{L^2}.$$

Recall that the kernel of d^{∇} is given by imaginary multiples of the identity, since ∇ is irreducible. Hence, it is injective as an operator $L^2_k(\mathfrak{u}(E))/i\mathbb{R}\mathrm{id}_E \to \Omega^1_{k-1}(\mathfrak{u}(E))$. In particular, since its symbol is injective, it has a bounded left-inverse G, say. Thus, we can write $\chi = G\mathrm{d}^{\nabla}\chi$ and get the estimate

$$||\chi||_{L^2_1} \leq c||\mathrm{d}^{\nabla}\chi||_{L^2}.$$

Now we use this to finish the proof. We have seen above that $0 = ||\mathrm{d}^{\nabla}\chi||^2_{L^2} + \langle \mathrm{d}^{\nabla}\chi, [A_1, \chi]\rangle_{L^2}$. Thus,

$$\begin{aligned}
||\mathrm{d}^{\nabla}\chi||^2_{L^2} &= |\langle(\mathrm{d}^{\nabla}\chi, [A_1, \chi]\rangle_{L^2}| \\
&\leq ||\mathrm{d}^{\nabla}\chi||_{L^2}||[A_1, \chi]||_{L^2} \qquad \text{by Cauchy-Schwartz} \\
&\leq ||\mathrm{d}^{\nabla}\chi||_{L^2}||A_1||_{L^4}||\chi||_{L^4} \qquad \text{by Hölder's inequality} \\
&\leq ||\mathrm{d}^{\nabla}\chi||_{L^2}||A_1||_{L^2_1}||\chi||_{L^2_1} \qquad \text{by Sobolev inequality} \\
&\leq c||\mathrm{d}^{\nabla}\chi||^2_{L^2}||A_1||_{L^2_1} \qquad \text{by the above estimate.}
\end{aligned}$$

Hence, if $||A_1||_{L^2_1} < C := \frac{1}{c}$, this gives $||\mathrm{d}^{\nabla}\chi||^2_{L^2} < ||\mathrm{d}^{\nabla}\chi||^2_{L^2}$, and thus we conclude

$$||\mathrm{d}^{\nabla}\chi||_{L^2} = 0,$$

and so by irreducibility χ has to be contained in $i\mathbb{R}\mathrm{id}_E$. Moreover, the integral of χ is zero, since it lies in the image of $(\mathrm{d}^{\nabla})^*$. From this we conclude that

$$\chi = 0.$$

That is, the differential $D_2F = (\mathrm{d}^{\nabla_1})^*\mathrm{d}^{\nabla}$ is surjective. Thus, if the norm of A_2 is sufficiently small, say less than some $\epsilon_2(\nabla_1) > 0$, the implicit function theorem guarantees the existence of a small gauge transformation which puts ∇_2 into

Coulomb gauge with respect to ∇_1. Note that the constant C only depends on ∇ and neither on ∇_1 nor ∇_2. $\square$

Corollary 4.4. *Every flat L^2_k-connection has a small neighbourhood on which a horizontal geodesic with flat endpoints is uniquely determined by its endpoints. In particular, an open subset of the moduli space of solutions to the harmonic map equations can be identified with a neighbourhood of the diagonal in the product of the moduli space of flat connections with itself.*

Proof. Let ∇ be a flat connection in $\mathcal{A}^*_k$ and let $\epsilon = \frac{1}{2}\min(\epsilon(\nabla), C)$, where $\epsilon(\nabla)$ is the constant from proposition 4.2 and C is the constant from theorem 4.3. We know from theorem 4.3 that we can cover the ball of radius ϵ about ∇, which we denote by $B_\epsilon(\nabla)$ by smaller balls on which horizontal geodesics are determined by their endpoints: For each flat connection $\nabla_1 \in B_\epsilon(\nabla)$ the theorem provides a small ball $B_{\epsilon_2(\nabla_1)}(\nabla_1)$ on which we can solve the Coulomb gauge equation. Let $\mathcal{F}_\epsilon = \{A_1 \in \Omega^1_{k-1} \mid F^{\nabla + A_1} = 0 \ ; \ (\mathrm{d}^\nabla)^* A_1 = 0 \ ; \ ||A_1||_{k-1} \le \epsilon \ \}$. That is, we have produced a covering

$$B_\epsilon(\nabla) \cap \mathcal{F}_\epsilon \subset \bigcup_{\nabla_1 \in B_\epsilon(\nabla) \cap \mathcal{F}_\epsilon} B_{\epsilon_2(\nabla_1)}(\nabla_1).$$

Now recall that the moduli space of flat connections is actually a finite-dimensional manifold, and so $\mathcal{F}_\epsilon$ is therefore a compact set. Thus, we can choose a finite subcover. Now put

$$U_\nabla = \bigcap_{i=1}^{N} B_{\epsilon_2(\nabla_1^i)}(\nabla_1^i),$$

where we only allow such connections for which $\nabla \in B_{\epsilon_2(\nabla_1^i)}(\nabla_1^i)$. This is an open neighbourhood of ∇ which, after shrinking, we may assume to be a ball. This then has the desired properties. $\square$

We have seen that geodesics become unique once they are short enough. Thus, the space of horizontal geodesics with flat endpoints locally looks like a product of a small ball in the moduli space of flat connections with itself. This subset is open in the harmonic map moduli space, since we know that near a flat connection the moduli space has dimension $4(n^2(g-1)+1)$, which equals twice the dimension of the moduli space of flat unitary connections.

Our argument shows that a subset in the moduli space of solutions to the harmonic map equations can be identified with a neighbourhood of the diagonal in the product of the moduli space of flat connections with itself. This is in general an open set in the moduli space of solutions to the harmonic map equations. Actually, we have shown that any two flat connections which lie in a small open ball determine a unique geodesic linking them *in this ball*. They could be linked by longer geodesics, which leave the neighbourhood and come back

4.2. **Conjugate Points and the Degeneracy Locus.** Recall that on a Riemannian manifold a Jacobi field is a tangent vector to the space of geodesics, i.e. given a geodesic γ , a Jacobi field is a vector field along γ of the form

$$J(t) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\gamma_s(t),$$

where γ_s is a family of geodesics such that $\gamma = \gamma_0$. Two points on a geodesic are *conjugate* if there exists a non-trivial Jacobi field along γ which vanishes at these two points.

Theorem 4.5. *Let $\gamma(t)$, $t \in [0,1]$ be a horizontal geodesic segment with flat endpoints, such that the connection $\gamma(1/2)$ is irreducible. If the endpoints of γ are conjugate, the hypersymplectic structure on the moduli space of harmonic maps is degenerate at γ.*

The endpoints of γ being conjugate means that there exists a one-parameter family $\gamma(s,t)$ of horizontal geodesics with flat endpoints, such that $\gamma(t) = \gamma(0,t)$ and the endpoints are gauge-equivalent for all s. Equivalently, there exists a Jacobi field along γ which is tangent to the gauge orbits through the endpoints of γ .

Proof. Let $\nabla^- = \gamma(0,0)$ and $\nabla^+ = \gamma(0,1)$. Consider the Jacobi field

$$Y(t) = \frac{\partial}{\partial s}|_{s=0}\gamma(s,t).$$

Then by assumption, $Y(0)$ and $Y(1)$ are tangent to the gauge orbit through $\gamma(0)$ and $\gamma(1)$ respectively. This means, there are Lie algebra elements $\xi^\pm \in \Gamma(\mathfrak{u}(E))$ such that

$$Y(0) = \mathrm{d}^{\nabla^-}\xi^- \qquad Y(1) = \mathrm{d}^{\nabla^+}\xi^+.$$

Tracking through our correspondence, we write $\nabla = \frac{1}{2}(\nabla^+ + \nabla^-)$, $\phi = \frac{1}{2}(\nabla^+ - \nabla^-) = \Phi - \Phi^*$. In other words

$$\gamma(s,t) = \nabla(s) + (2t-1)\phi(s),$$

and so $\nabla = \nabla(0)$ and $\phi = \phi(0)$. We write $\nabla(s) = \nabla + A(s)$. In this notation

$$Y(0) = \dot{A} - \dot{\phi} \qquad Y(1) = \dot{A} + \dot{\phi}.$$

This gives

$$\begin{aligned} \dot{A} - \dot{\phi} &= \mathrm{d}^\nabla\xi^- - [\Phi - \Phi^*, \xi^-] \\ \dot{A} + \dot{\phi} &= \mathrm{d}^\nabla\xi^+ + [\Phi - \Phi^*, \xi^+]. \end{aligned}$$

It follows that

$$\begin{aligned} 2\dot{A} &= \mathrm{d}^\nabla(\xi^+ + \xi^-) + [\Phi - \Phi^*, \xi^+ - \xi^-] \\ 2\dot{\phi} &= \mathrm{d}^\nabla(\xi^+ - \xi^-) + [\Phi - \Phi^*, \xi^+ + \xi^-]. \end{aligned}$$

This means, that in complex co-ordinates $2(\dot{A}, \dot{\phi})$ is represented by the point

$$\begin{aligned} (2\dot{A}^{0,1}, 2\dot{\phi}^{1,0}) &= (\bar{\partial}^\nabla(\xi^+ + \xi^- - [\Phi^*, \xi^+ - \xi^-], \partial^\nabla(\xi^+ - \xi^-) + [\Phi, \xi^+ + \xi^-]) \\ &= (\bar{\partial}^\nabla(\xi^+ + \xi^-, [\Phi, \xi^+ + \xi^-]) + (-[\Phi^*, \xi^+ - \xi^-], \partial^\nabla(\xi^+ - \xi^-)) \\ &= X^{\xi^+ + \xi^-} + SX^{\xi^+ - \xi^-}. \end{aligned}$$

It is then easy to check that $\xi = \xi^+ - \xi^-$ defines an element of the degeneracy space at (∇, ϕ). Indeed, since $(\dot{A}, \dot{\phi})$ is tangent to the moduli space as well as $X^{\xi^+ + \xi^-}$, the vector SX^ξ has to be tangent to the solution space, too. This means it solves the linearised harmonic map equations, which in this situation precisely yield the degeneracy equation for ξ.

This shows that if the endpoints of a geodesic corresponding to a solutions to the harmonic map equations are *conjugate*, then the hypersymplectic structure is degenerate at this point of the moduli space. In this way, we see that flat

connection contained in the image of the *cut locus* of the Riemannian manifold $\mathcal{A}_k^*/\mathcal{G}_{k+1}^*$ under the exponential map will also be contained in the degeneracy locus of the hypersymplectic structure.

$\square$

On our nice open set the converse of this theorem is also true.

Proposition 4.6. *The hypersymplectic structure on the open set, on which geodesics are determined by their flat endpoints, is non-degenerate.*

Proof. Let (∇, ϕ) be a solution and suppose $||\mathrm{ad}(\phi)||_{L^2}^2 < \lambda_1(\nabla)$, where $\lambda_1(\nabla)$ is the smallest non-zero eigenvalue of $(\mathrm{d}^\nabla)^*\mathrm{d}^\nabla$ acting on sections of $\mathfrak{u}(E)$. Assume further that $\eta \in L_2^2(\mathfrak{u}(E))$ is a solution to the elliptic degeneracy equation, i.e.

$$(\mathrm{d}^\nabla)^*\mathrm{d}^\nabla\eta - (\mathrm{ad}(\phi))^*\mathrm{ad}(\phi)(\eta) = 0.$$

Taking the inner product with η yields

$$||\mathrm{d}^\nabla\eta||_{L^2}^2 = ||[\phi, \eta]||_{L^2}^2.$$

This gives

$$\begin{array}{rcl} \lambda_1(\nabla)||\eta||_{L^2}^2 & \leq & ||\mathrm{d}^\nabla\eta||_{L^2}^2 \\ & = & ||[\phi, \eta]||_{L^2}^2 \\ & \leq & ||\mathrm{ad}(\phi)||_{L^2}^2||\eta||_{L^2}^2 \\ & < & \lambda_1(\nabla)||\eta||_{L^2}^2. \end{array}$$

Therefore, η has to vanish in $\mathrm{Lie}(\mathcal{G}^*)$ and the point (∇, ϕ) does not belong to the degeneracy locus. $\square$

5. Conclusion and Final Comments

As we have already remarked, with minor modifications the proofs above still work if we replace $\mathrm{U}(n)$ by an arbitrary compact Lie group G. We have shown that there exists a hypersymplectic structure on a suitable neighbourhood of the moduli space of irreducible flat G-connections inside the moduli space of solutions to the gauge theoretic harmonic map equations.

We have seen that with a solution of the equations we can associate in a natural way a pair of flat unitary connections over the compact Riemann surface M. Locally, the two flat connections are *trivial*, and hence they have to be gauge equivalent by a local gauge transformation. That is, in a parallel trivialisation (with respect to ∇^+ say), their difference, which equals 2ϕ, has to be of the form $u^{-1}\mathrm{d}u$, for a smooth map $u : U \subset M \to G$, which as a consequence of the holomorphicity of Φ is *harmonic*. This construction produces a harmonic section of the flat bundle of groups associated with the action of $G\times G$ on G given by $(a, b).u = a^{-1}ub$ equipped the product connection $(\nabla_\theta^+, \nabla_\theta^-)$. In other words, the moduli space parametrises harmonic sections of flat $G \times G$-bundles. For zero Higgs field the two connections coincide and there are no non-trivial harmonic sections due to the irreducibility of the flat connection. Thus, we have the moduli space of irreducible flat connections embedded in a natural way.

In fact, we have seen in theorem 4.3 that we may naturally think of this open set inside the moduli space of solutions to the harmonic map equations as the *paracomplexification* of the moduli space of flat G-connections. The local product structure S exhibits this open set as a neighbourhood of the diagonal inside the

product of the moduli space of flat G-connections with itself. This is the split signature analogue of the fact that the Higgs bundle moduli space with the complex structure J is the moduli space of flat $g^{\mathbb{C}}$-connections. Note that $G \times G$ is the paracomplexification of G, and so we see that the analogy is actually very close of the case of Higgs bundles: With respect to the paracomplex structure S the open set inside the moduli space describes flat paracomplex connections.

Acknowledgements. This work is based on parts of the author's DPhil thesis [7]. Special thanks are due to Prof. Andrew Dancer for suggesting this DPhil project and for his support, guidance and encouragement over the past three years. The author is grateful to Prof. Nigel Hitchin and Prof. Lionel Mason for a number of useful remarks. Most of this work was carried out at the Mathematical Institute of the University of Oxford supported by a DPhil studentship of the Engineering and Physical Sciences Research Council. The author also wishes to thank the University of Münster and the SFB 878 "Groups, Geometry and Actions" for research support.

References

[1] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308:523–615, 1982.

[2] A.S. Dancer and A. Swann. Hypersymplectic manifolds. In *Recent developments in pseudo-Riemannian geometry*, ESI Lect. Math. Phys.,, pages 97–111, Zürich, 2008. Eur. Math. Soc.

[3] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1990.

[4] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[5] N. J. Hitchin. Harmonic maps from a 2-torus to the 3-sphere. *J. Differential Geom.*, 31(3):627–710, 1990.

[6] N. J. Hitchin. Hypersymplectic quotients. *Acta Acad. Sci. Tauriensis*, 124:169–180, 1990.

[7] M. Röser. *The ASD Equations in Split Signature and Hypersymplectic Geometry.* PhD thesis, University of Oxford, 2012.

[8] C. T. Simpson. Nonabelian Hodge theory. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 747–756, Tokyo, 1991. Math. Soc. Japan.

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, Münster, Germany

E-mail address: mroes_02@ uni-muenster.de